Dynamics of Strongly Interacting Fermi Gases of Atoms in a Harmonic Trap

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Dynamics of strongly interacting trapped dilute Fermi gases is investigated at zero temperature. As an example of application we consider the expansion of the cloud of fermions initially confined in an anisotropic harmonic trap, and study the equation of state dependence of the radii of the trapped cloud and the collective oscillations in the vicinity of a Feshbach resonance.

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The newly created ultracold trapped Fermi gases with tunable atomic scattering length [1-10] in the vicinity of a Feshbach resonance offer the possibility to study highly correlated many-body systems including the cross-over from the Bardeen-Cooper-Schrieffer (BCS) phase to the Bose-Einstein condensate (BEC) of molecules [1,11-15].

In this letter we report our investigation of the dynamics of the strongly interacting dilute Fermi gas (dilute in the sense that the range of interatomic potential is small compared with inter-particle spacing) at zero temperature. As an example of application we consider the expansion of the cloud of ^6Li atoms initially confined in an anisotropic harmonic trap, study the equation of state dependence of the radii of the trapped cloud and the collective oscillations near a broad Feshbach resonance at a magnetic field $B = 820 \pm 3$G [16-18].

We consider a Fermi gas comprising a 50-50 mixture of two different states confined in a harmonic trap $V_{ext}(\vec{r}) = (m/2)(\omega_\perp^2(x^2+y^2)+\omega_z^2 z^2)$. The s-wave scattering length between the two fermionic species is negative, $a < 0$.

Our starting point is the single equation approach to the time-dependent density-functional theory [19]. The basic of this strategy is to construct the following equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V_{ext}\Psi + V_{xc}\Psi \quad (1)$$

that yields the same $n(\vec{r},t) =\mid \Psi(\vec{r},t) \mid^2$ as the original many-fermions system. The dynamics of the system is controlled by an effective single-particle potential $V_{xc}(\vec{r},t)$. The central problem is the approximation for the xc potential. This is in general a nonlocal functional of the density that depends on the history of the system (on the behavior of the density at times $t' < t$).

The simplest approximation is to ignore nonlocality in space and retardation in time. This leads to the adiabatic local density approximation

$$V_{xc}(\vec{r},t) = [\frac{\partial n\epsilon(n)}{\partial n}]_{n=n(\vec{r},t)} \quad (2)$$

where ϵ is the ground state energy per particle of the homogeneous system and n is the density. The right hand side of Eq.(2) is the local density approximation for the ground - state xc potential, but it evaluated at the time-dependent density. Notice that in the above equation n is the total density of the gas given by the sum of the two spin component.

The adiabatic local density approximation is a remarkably good approximation if the energy gap is much larger than the oscillator energies $\hbar\omega_z$, $\hbar\omega_\perp$ [20,21]. It is expected that

this condition is satisfied for small temperature [20,21]. Here we notice Refs.[22-24] who argue that the ground state of the mixture of two species of fermions with different densities (mass) contains both a superfluid and a normal Fermi liquid. We do not consider this asymmetrical mixture in the letter.

The ground state energy per particle, $\epsilon(n)$, in the low-density regime, $k_F \mid a \mid\ll 1$, can be calculated using an expansion in power of $k_F \mid a \mid$

$$\epsilon(n) = 2E_F(\frac{3}{10}-\frac{1}{3\pi}k_F \mid a \mid +0.055661(k_F \mid a \mid)^2-0.00914(k_F \mid a \mid)^3-0.018604(k_F \mid a \mid)^4+...), \tag{3}$$

where $E_F = \frac{\hbar^2 k_F^2}{2m}$ and $k_F = (3\pi^2 n)^{1/3}$. The expansion (3) is valid for $3D$. For the case of dimensions $d < 3$, it is known that the quantum-mechanical two-body t-matrix vanishes at low energies [25]. The first term in Eq.(3) is the Fermi kinetic energy, the second term corresponds to the mean-field prediction [26], the next two terms were first considered in Refs.[27,28] and Ref.[29], respectively.

In the $a \to -\infty$ limit (the Bertsch many-body problem, quoted in Ref.[30]) $\epsilon(n)$ is proportional to that of the non-interacting Fermi gas

$$\epsilon(n) = (1+\beta)\frac{3}{10}\frac{\hbar^2 k_F^2}{m}, \tag{4}$$

where a universal parameter β [9] is negative and $\mid \beta \mid< 1$ [30-32].

We also consider the following approximations for $\epsilon(n)$:

$$\epsilon(n) = E_F(\frac{3}{5} - \frac{(2/(3\pi))k_F \mid a \mid}{1+(6/(35\pi))(11-2\ln 2)k_F \mid a \mid}), \tag{5}$$

and

$$\epsilon(n) = E_F(\frac{3}{5} - 2\frac{\delta_1 k_F \mid a \mid +\delta_2 (k_F \mid a \mid)^2}{1+\delta_3 k_F \mid a \mid +\delta_4 (k_F \mid a \mid)^2}), \tag{6}$$

where $\delta_1 = 0.106103$, $\delta_2 = 0.187515$, $\delta_3 = 2.29188$, $\delta_4 = 1.11616$.

While Eq.(5) [31] reproduces first three terms of expansion (3) in low-density regime and approximately valid in unitary limit, $\beta = -0.67$, Eq.(6) reproduces first four terms of expansion (3) in low-density regime and in unitary limit, $k_F a \to -\infty$, reproduces exactly results of the recent Monte Carlo calculations [32], $\beta = -0.56$.

It can be proved [33] that every solution of the equation

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V_{ext}\Psi + \frac{\partial (n\epsilon(n))}{\partial n}\Psi, \tag{7}$$

is a stationary point of an action corresponding to the Lagrangian density

$$\mathcal{L}_0 = \frac{i\hbar}{2}(\Psi\frac{\partial\Psi^*}{\partial t} - \Psi^*\frac{\partial\Psi}{\partial t}) + \frac{\hbar^2}{2m} \mid \nabla\Psi \mid^2 + \epsilon(n)n + V_{ext}n, \tag{8}$$

which for $\Psi = e^{i\phi(\vec{r},t)}n^{1/2}(\vec{r},t)$ can be rewritten as

$$\mathcal{L}_0 = \hbar\dot{\phi}n + \frac{\hbar^2}{2m}(\nabla\sqrt{n})^2 + \frac{\hbar^2}{2m}n(\nabla\phi)^2 + \epsilon(n)n + V_{ext}n. \tag{9}$$

The only difference from equations holding for bosons [33,34] is given by density dependence of $\epsilon(n)$. We do not consider three-body recombinations, since these processes play an important role near p-wave two-body Feshbach resonance [35].

Let us first discuss the expansion of the fermionic superfluid in the $a \to -\infty$ limit, Eq.(4). In the hydrodynamic approximation (neglecting quantum pressure term, $\frac{\hbar^2}{2m}(\nabla\sqrt{n})^2$, in Eq. (9)) the corresponding Euler-Lagrange equation admit the simple scaling solution, $n(\vec{r},t) = n_0(x_i/b_i(t))$ [20]. We note here that the hydrodynamic behavior of a cold Fermi gas [9] is not in general direct experimental evidence for superfluidity [36-38].

We take into account the quantum pressure by finding the optimal ground state energy [39]

$$\frac{E_0}{N} = \max_{\gamma_x,\gamma_y,\gamma_z}[\sum_{i=1}^{3}\frac{\hbar\omega_i}{2}\sqrt{\gamma_i} + \frac{3^{4/3}}{4}(1+\beta)^{1/2}N^{1/3}\prod_{i=1}^{3}(\sqrt{1-\gamma_i}\omega_i)^{1/3}]. \tag{10}$$

In this case the scaling parameters obey the following equations

$$\ddot{b}_i - \frac{\omega_i^2\gamma_i}{b_i^3} = \frac{\omega_i^2(1-\gamma_i)}{b_i\prod_{i=1}^{3}b_i^{2/3}}, \tag{11}$$

at $t=0$ $b_i(0)=1$ and $\dot{b}_i(0)=0$.

The predictions of Eqs.(11) for aspect ratio, $\omega_z\sqrt{1-\gamma_z}b_\perp/(\omega_\perp\sqrt{1-\gamma_\perp}b_z(t))$, are reported in Fig.1 and show that the effect of inclusion of the quantum pressure term on the expansion of superfluid is about 1%. For the reminder of this letter we will use the hydrodynamic approximation.

Now we consider a general time-dependent harmonic trap, $V_{ext}(\vec{r},t) = (m/2)\sum_{i=1}^{3}\omega_i^2(t)x_i^2$, and a general $\epsilon(n)$. A suitable trial function can be taken as $\phi(\vec{r},t) = \chi(t) + (m/(2\hbar)\sum_{i=1}^{3}\eta_i(t)x_i^2$, $n(\vec{r},t) = n_0(x_i/b_i(t))/\prod_j b_j$. With this ansatz, the Hamilton principle, $\delta\int dt\int\mathcal{L}_0 d^3r = 0$, gives the following equations for the scaling parameters b_i

$$\ddot{b}_i + \omega_i^2(t)b_i - \frac{\omega_i^2}{b_i}\frac{\int[n^2 d\epsilon(n)/dn]_{n=n_0(\vec{r})/\prod_j b_j}d^3r}{\int[n^2 d\epsilon(n)/dn]_{n=n_0(\vec{r})}d^3r}\prod_j b_j = 0, \tag{12}$$

where $b_i(0) = 1$, $\dot{b}_i(0) = 0$ and $\omega_i = \omega_i(0)$ fix the initial configuration of the system, corresponding to the density $n_0(\vec{r})$.

The release energy which corresponds to an integral of motion of Eq.(12) is expressed by

$$E_{rel} = \frac{1}{N}[\frac{1}{2}\frac{\dot{b}_i^2}{\omega_i^2}\int n_0^2(\vec{r})\frac{d\epsilon(n_0)}{dn_0}d^3r + \int n_0\epsilon(n_0/\prod_j b_j)d^3r], \tag{13}$$

and for the case of $\epsilon(n) \propto n^\gamma$

$$E_{rel} = \frac{2\mu}{5\gamma+2}[\frac{\gamma}{2}\frac{\dot{b}_i^2}{\omega_i^2} + \frac{1}{\prod_j b_j^\gamma}],$$

where μ is the chemical potential.

Expanding Eqs.(12) around equilibrium ($b_i = 1$) we get in the case of anisotropic trapping ($\omega_x = \omega_y = \omega_\perp$, $\omega_z/\omega_\perp = \lambda$) the following result for the frequency of the radial compression mode

$$\omega_{rad} = \frac{\omega_\perp}{\sqrt{2}}[4 + 2\kappa + 3\lambda^2 + \kappa\lambda^2 + \sqrt{(4 + 2\kappa + 3\lambda^2 + \kappa\lambda^2)^2 - 4(10 + 6\kappa)\lambda^2}]^{1/2}, \tag{14}$$

where $\kappa = \int n_0^3 d^2\epsilon/(dn_0^2)d^3r/\int n_0^2 d\epsilon/(dn_0)d^3r$. For an elongated trap, $\lambda \ll 1$, we can rewrite Eq.(14) as

$$\omega_{rad} \approx \omega_\perp\sqrt{4 + 2\kappa}. \tag{15}$$

Note that Eq.(14) for the case of $\epsilon(n) \propto n^\gamma$ was considered in several papers [40].

In Fig. 2 we present the calculations of ω_{rad} using two approximations, Eq.(5) and Eq.(6), for the equation of state $\epsilon(n)$ (to calculate the ground-state density we have used a highly accurate variational approach of Ref.[41]). The curves explicitly show the nonmonotonic behavior of ω_{rad} in the agreement with a schematic interpolation of Ref.[42]. It can be seen from Fig. 2 that the difference between two approximations, Eq.(5) and Eq.(6), is less than 0.7%.

Our calculated results for the axial cloud size of strongly interacting ^6Li atoms as a function of the magnetic field strength B are compared with the recent experimental data [10] in Fig. 3. This comparison shows that although both approximations, Eq.(5) and Eq.(6), give a reasonable agreement with experimental data, the equation of state from Eq.(6) leads to the better description of the experimental curve. We have used the data from Ref.[17] to convert a to B.

We note here that in general a Feshbach resonance may lead to the density dependence of the effective interaction (for bosons cases see, for example, [43,44]).

In conclusion, we have considered the expansion of the cloud of initially confined ^6Li atoms and studied the equation of state dependence of the radii of trapped cloud and collective oscillations near the broad Feshbach resonance at $B = 820 \pm 3$G. It is shown a non monotonic behavior of the radial compression mode frequency and demonstrated that an important test of the equation of state can be obtained from the study of the radii of trapped cloud in regimes now available experimentally.

Note added: A recent paper by the Duke University group [45] reports on measurements of the radial compression mode frequencies. Our calculations in a very good agreement with these experimental data on the BCS side.

While this work was being prepared for publication, two preprints [46,47] appeared in which the authors consider collective modes and the expansion of a trapped superfluid Fermi gas in the BCS-BEC crossover. For the negative scattering length case, their results are in perfect agreement with ours.

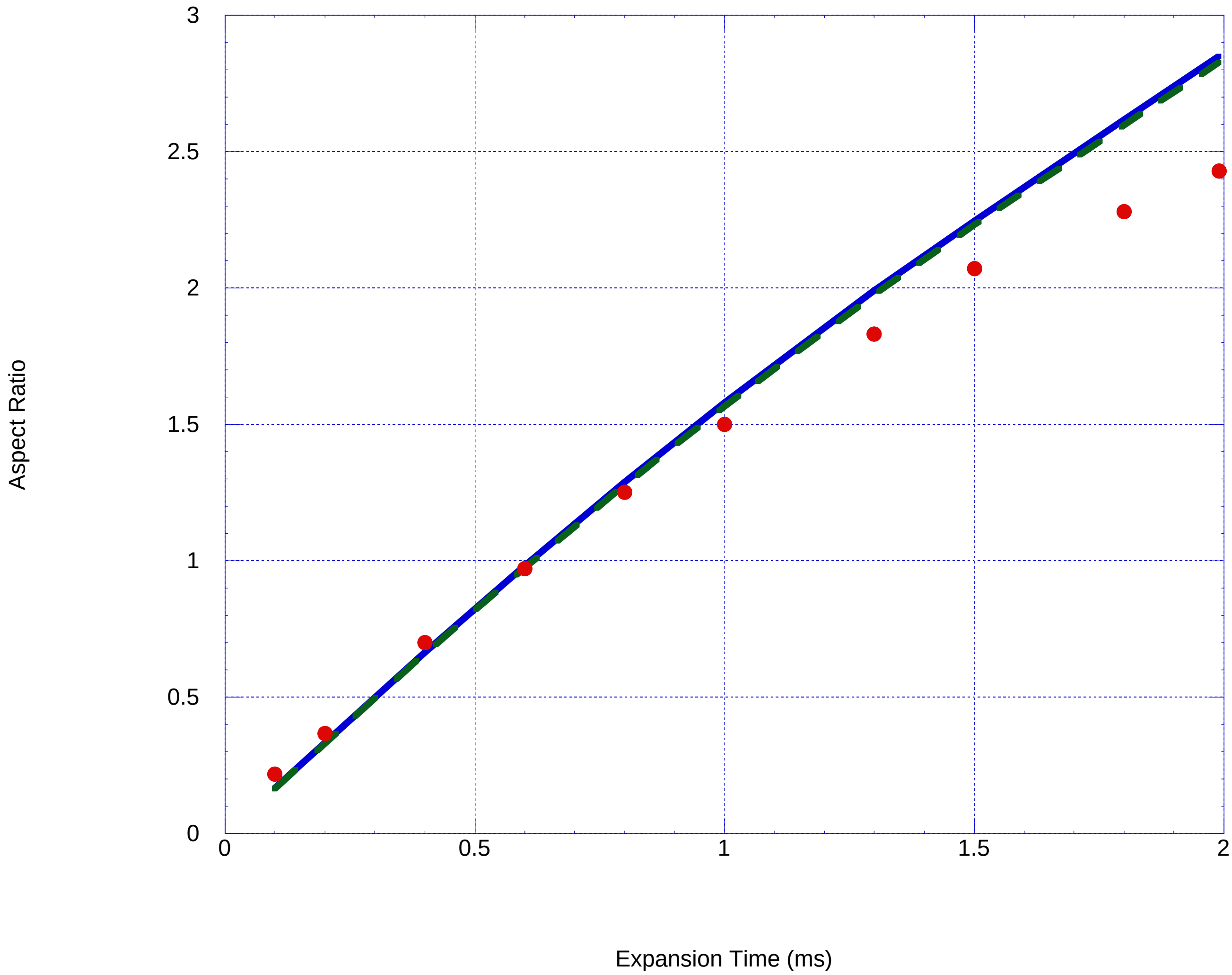

Fig. 1. Aspect ratio of the cloud of the $N = 7.5 \times 10^4\ {}^6Li$ atoms as a function of time after release from the trap ($\omega_\perp = 2\pi \times 6605\text{Hz}$, $\omega_z = 2\pi \times 230\text{Hz}$). The circular dots indicate experimental data from the Duke University group [9]. The solid line and the dashed line represent theoretical calculations in the unitary limit ($a \to -\infty$) including the quantum pressure term and in the hydrodynamic approximation, respectively.

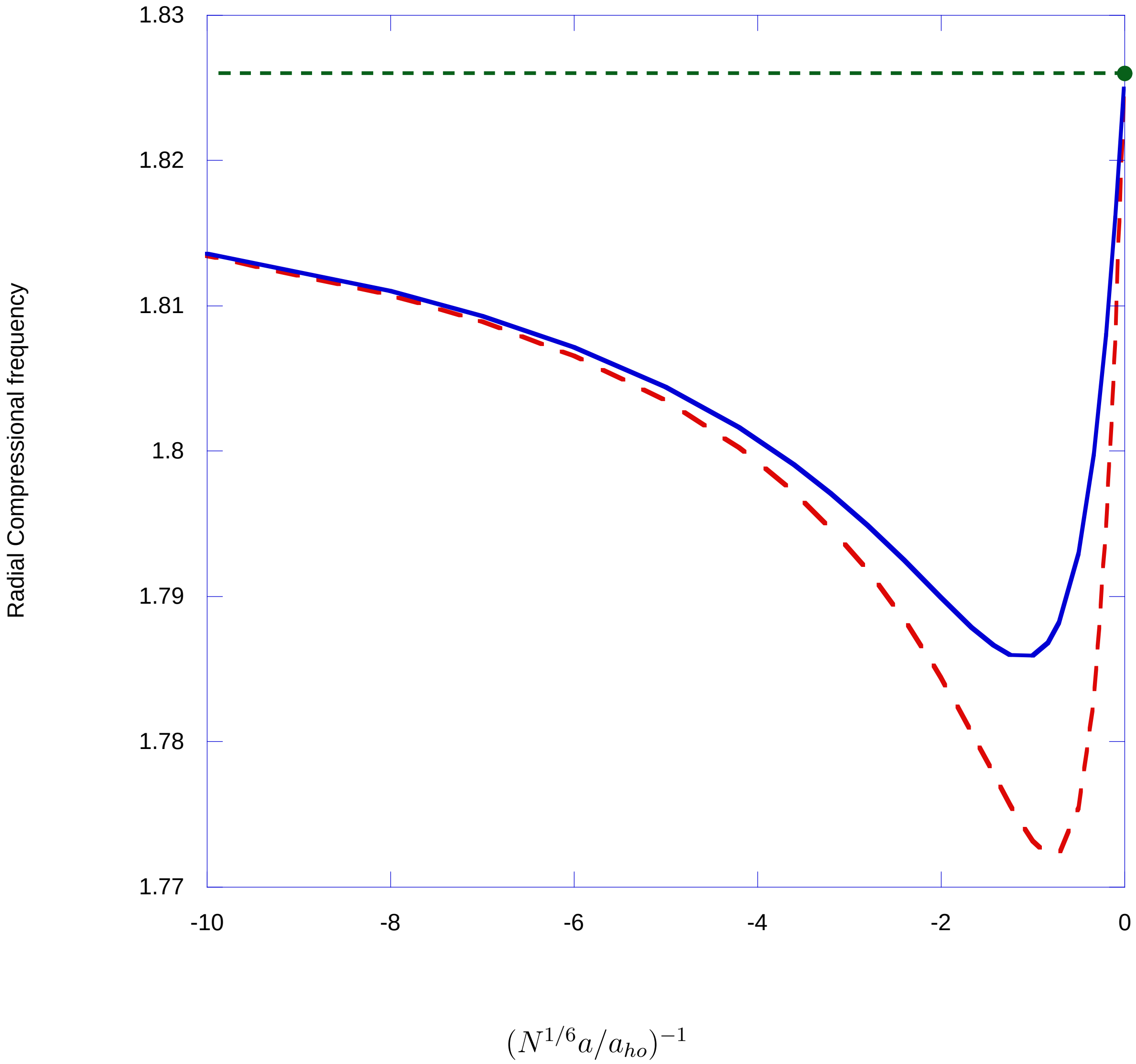

Fig. 2. Radial compressional frequency in unit of $\omega_\perp$ as a function of the dimensional parameter $(N^{1/6}a/a_{ho})^{-1}$. In the unitary limit, $a \rightarrow -\infty(\bullet)$, one expect $\omega/\omega_\perp = \sqrt{10/3} \approx 1.826$. The solid line and the dashed line represent the results of theoretical calculations using equations of state Eq.(6) and Eq.(5) respectively.

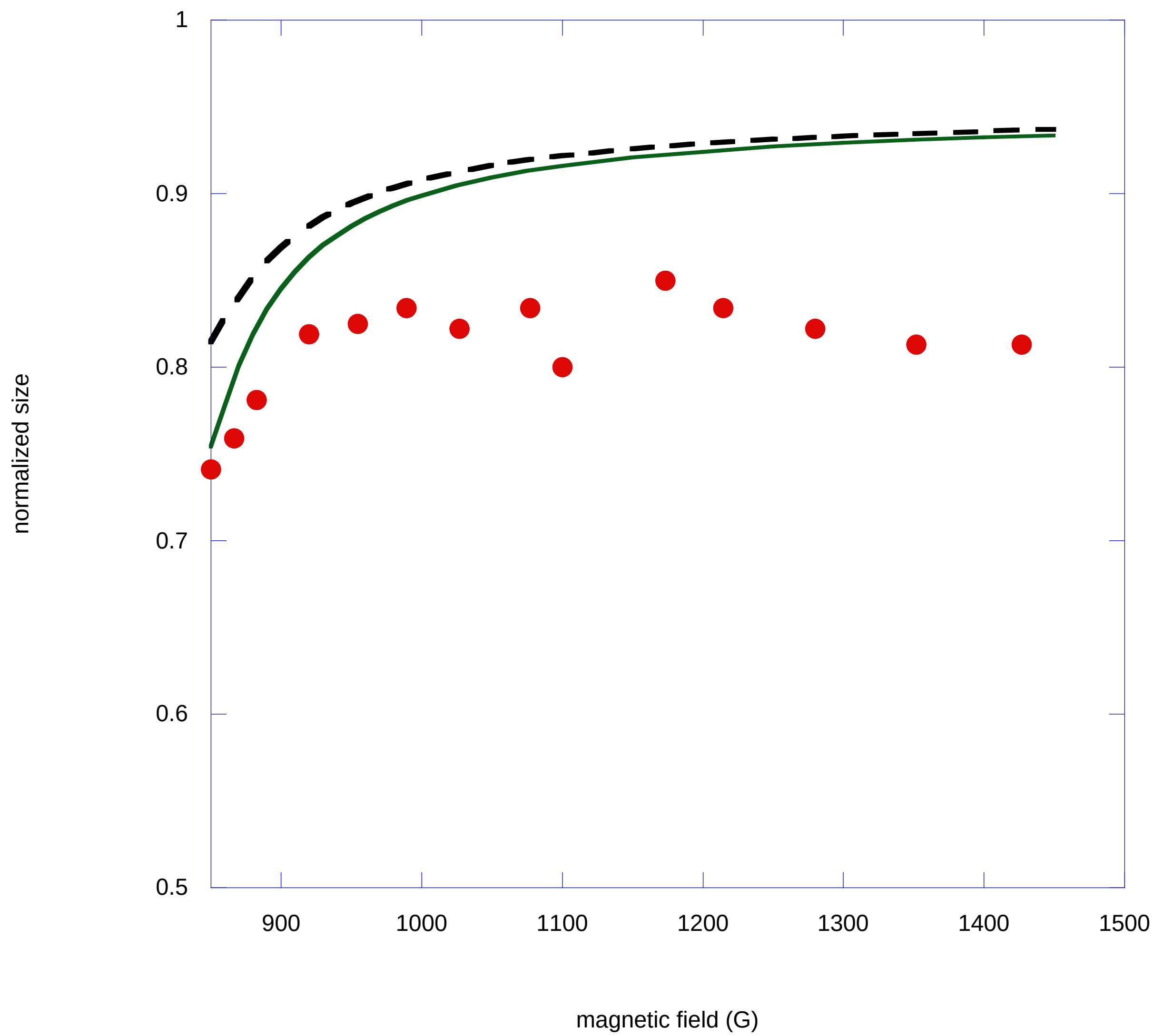

Fig. 3. Axial cloud size of strongly interacting ^6Li atoms after normalization to a non-interacting Fermi gas with $N = 4 \times 10^5$ atoms as a function of the magnetic field B. The trap parameters are $\omega_\perp = 2\pi \times 640$Hz, $\omega_z = 2\pi(600B/kG + 32)^{1/2}$Hz. The solid line, dashed line and circular dots represent the results of theoretical calculations using equations of state Eq. (6), Eq. (5) and experimental data from the Innsbruck group [10], respectively.

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